Data analysis

جامعة باجي مختار -عنابة
كلية العلوم الاقتصادية وعلوم التسيير
FACULTY OF ECONOMICS AND MANAGEMENT
Badji Mokhtar – Annaba University

Table of contents

I Calculations in Matrix Algebra

1. Course 1 : Calculations in Matrix Algebra

1.1. Matrix Overview

🔍 *Definition*

A matrix is a rectangular array of numbers arranged in rows and columns. For example, a matrix A of size $m \times n$ has m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

✎ *Note:Vector Representation*

A vector is a special type of matrix with only one column (column vector) or one row (row vector). For example, the column vector X , representing variables in regression, is expressed as :

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Types of Matrices

- ***Row Matrix:*** A matrix with only one row ($1 \times n$).
  $$A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}.$$
- ***Column Matrix*** A matrix with only one column ($m \times 1$).
  $$B = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}.$$
- ***Square Matrix:*** A matrix with the same number of rows and columns $(n \times n)$.
  $$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$
- ***Zero Matrix:*** A matrix where all elements are zero.
  $$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
- ***Diagonal Matrix***: A square matrix where all the non-diagonal elements are zero.
  $$E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$
- ***Identity Matrix:*** A square matrix with ones on the diagonal and zeros elsewhere.
  $$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- ***triangular matrix*** is a type of square matrix where either all the entries below or above the main diagonal are zero. There are two types of triangular matrices:
- In an ***upper triangular matrix***, all the elements below the main diagonal are zero. Formally, a matrix A is upper triangular if:

  $a_{ij} = 0 \quad \text{for } i > j.$

  $$F = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$
- In ***a lower triangular matrix***, all the elements above the main diagonal are zero. Formally, a matrix A is lower triangular if: $a_{ij} = 0 \quad \text{for } i < j.$

  $$G = \begin{bmatrix} 7 & 0 & 0 \\ 8 & 9 & 0 \\ 10 & 11 & 12 \end{bmatrix}.$$

Definition: Equality of Matrices

Two matrices A and B are said to be equal, denoted $A = B$, if:

- They have the same number of rows and columns.

For every element a_{ij} in matrix A and the corresponding element b_{ij} in matrix B, it holds that $a_{ij} = b_{ij}$ for all i and j.

Let:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Both matrices A and B are 2×3, and all corresponding elements match:

$A = B.$

1.2. Basic matrix operations

Addition and substraction of matrices

Two matrices of the same size can be added element-wise.

$$C = A + B \implies c_{ij} = a_{ij} + b_{ij}.$$

Properties:

- ***Commutative*** Law:

A + B = B + A.

- ***Associative*** Law:

A + (B + C) = (A + B) + C = A + B + C.

- A + (-A) = 0 (where –A is the matrix composed of –aij as elements and 0 is a matrix with all elements are equal to 0).

Matrix Multiplication

- A matrix can be multiplied by a scalar (a single number).

$B = kA \implies b_{ij} = k \cdot a_{ij}.$

- The product of two matrices A (size $m \times n$) and B (size $n \times p$) is defined as:
  $C = AB \implies c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$

Properties :

- ***Distributive Property of Scalar Multiplication:*** For any scalar k and matrices A and B,

$k(A + B) = kA + kB.$

- ***Distributive Property of Scalars over Matrices:*** For any scalars k and g and matrix A,

$(k + g)A = kA + gA.$

- ***Scalar Multiplication with Matrix Product:*** For any scalar k and matrices A and B,

$k(AB) = (kA)B = A(kB).$

- ***Associative Property of Scalar Multiplication:*** For any scalars k and and matrix A,

$k(gA) = (kg)A.$

Example:Matrix calcul

See exercise 1 of TD

Definition: Matrix transpose

The transpose of a matrix A is obtained by flipping it over its diagonal.

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}.$$

Definition: Determinant of a matrix

A scalar value that can be computed from the elements of a square matrix, denoted as $\det(A)$ or $|A|$.

The determinant of a square matrix A is a scalar value that can be computed from its elements.

- For a 2×2 matrix:
  $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad \det(A) = ad - bc.$$
- For larger matrices, the determinant can be computed using various methods, including cofactor expansion; such as a 3×3 matrix:
  $$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \Rightarrow \quad \det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix},$$
  which expands to:
  $$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg).$$

Example:Calcul of determinant

Consider the 3×3 matrix:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 1 & 4 & 2 \end{pmatrix}.$$

To find $\det(A)$, we use cofactor expansion along the first row:

$$\det(A) = 2 \begin{vmatrix} 1 & 0 \\ 4 & 2 \end{vmatrix} - 0 \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}.$$

Simplifying further:

$$\det(A) = 2(1 \times 2 - 0 \times 4) + 1(3 \times 4 - 1 \times 1) = 2(2) + 1(12 - 1) = 4 + 11 = 15.$$

Definition: Matrix Inversion

The inverse of a matrix satisfies the equation:

$$AA^{-1} = A^{-1}A = I,$$

where I is the identity matrix.

The inverse of a matrix A of size $n \times n$ can be calculated using the formula:

$$A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A),$$

where $\det(A)$ is the determinant of A and $\mathrm{adj}(A)$ is the adjugate matrix (the transpose of the ***cofactor matrix of A***).

Example:Calcul of the inverse

Consider the following 3×3 matrix A:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 1 & 4 & 2 \end{pmatrix}.$$

1-We previously computed the determinant of A:

$$\det(A) = 15.$$

2-To compute the adjugate, we need to calculate the cofactors of each element of A. The cofactor C_{ij} is given by:

$$C_{ij} = (-1)^{i+j} \cdot \det(M_{ij}),$$

where M_{ij} is the minor matrix formed by deleting the i-th row and j-th column of A.

- For element a_{11} = 2, the minor matrix is:

$$M_{11} = \begin{pmatrix} 1 & 0 \\ 4 & 2 \end{pmatrix}, \quad \det(M_{11}) = (1 \times 2 - 0 \times 4) = 2.$$

Thus, $C_{11} = (+1) \times 2 = 2.$

- For element $a_{12} = 0$, the minor matrix is:

$$M_{12} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}, \quad \det(M_{12}) = (3 \times 2 - 0 \times 1) = 6.$$

Thus, $C_{12} = (-1) \times 6 = -6.$

- For element $a_{13} = 1$, the minor matrix is:

$$M_{13} = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, \quad \det(M_{13}) = (3 \times 4 - 1 \times 1) = 11.$$

We repeat this process for all elements of A, resulting in the cofactor matrix:

3-The adjugate matrix is the transpose of the cofactor matrix:

$$\text{Adjugate Matrix} = \begin{pmatrix} 2 & 4 & -1 \\ -6 & 3 & 3 \\ 11 & -8 & 2 \end{pmatrix}.$$

4-Finally, the inverse of A is:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A) = \frac{1}{15} \cdot \begin{pmatrix} 2 & 4 & -1 \\ -6 & 3 & 3 \\ 11 & -8 & 2 \end{pmatrix}.$$