Data analysis

Chapter2 :
Linear Transformations and Eigenvalues and eigvectors

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1. Course 2: Linear Transformations, Eigenvalues and eigvectors

This course introduces key concepts in linear algebra, focusing on linear transformations, eigenvalues, and eigenvectors. Students will learn how linear transformations map vectors between spaces while preserving vector operations. The course also covers matrix representations, systems of linear equations, and the use of eigenvalues and eigenvectors for simplifying matrix operations.

1.1. Definition of Linear Transformations

Definition of Linear Transformations

A linear transformation is ***a function*** that maps vectors from one vector space to another while preserving the operations of vector addition and scalar multiplication. Formally, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and scalar c, the following must hold:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
2. $T(c\mathbf{u}) = cT(\mathbf{u})$.

Representation of Linear Transformations Using Matrices

Any linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented by an $m \times n$ matrix A, where $T(\mathbf{x}) = A\mathbf{x}$, for any vector $\mathbf{x} \in \mathbb{R}^n$.

Definition: Systems of Linear Equations

A system of linear equations can be represented in matrix form. For example, consider the system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}.$$

This can be written compactly as:

$$AX = B,$$

where A is the coefficient matrix, X is the ***vector of unknowns***, and B is the ***vector of constants***.

✎ *Note:Solving Using Matrix Inverses*

If the ***matrix A is invertible***, we can solve the system $AX = B$ by multiplying both sides by A^{-1}, giving:

$X = A^{-1}B.$

1.2. Example in Economics (Marketing):

Consider a scenario in marketing where a company decides to apply a transformation matrix to its advertising strategy across two platforms: social media and traditional media. Suppose the effect of increasing the budget for these two platforms on brand awareness and customer engagement can be represented as a linear transformation. Let the vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ represent the investments in social media (x_1) and traditional media (x_2), and the matrix A represents the transformation of these investments into outcomes like brand awareness and customer engagement:

$$A = \begin{pmatrix} 0.7 & 0.2 \\ 0.5 & 0.8 \end{pmatrix}.$$

The new vector $T(\mathbf{x}) = A\mathbf{x}$ will give the transformed values, representing the increased outcomes for awareness and engagement.

For instance, if $\mathbf{x} = \begin{pmatrix} 100 \\ 50 \end{pmatrix}$, the transformation results in:

$$T(\mathbf{x}) = \begin{pmatrix} 0.7 & 0.2 \\ 0.5 & 0.8 \end{pmatrix}\begin{pmatrix} 100 \\ 50 \end{pmatrix} = \begin{pmatrix} 80 \\ 90 \end{pmatrix}.$$

This suggests that increasing the budget to 100 units in social media and 50 units in traditional media leads to an 80% increase in brand awareness and 90% in customer engagement.

1.3. Eigenvalues and Eigenvectors

🔑 *Definition: Eigenvalue:*

Eigenvalues and eigenvectors are fundamental concepts in linear algebra with applications in fields like data science. They simplify matrix transformations, making complex systems easier to analyze.

To compute ***Eigenvalues*** λ:

We write the equation $A\mathbf{v} = \lambda\mathbf{v}$ as $(A - \lambda I)\mathbf{v} = 0$, where I is the ***identity matrix***. For non-zero solutions $\mathbf{v}$, the determinant

$$\det(A - \lambda I) = 0.$$

This leads to ***a polynomial equation*** in λ called the $characteristic\ polynomial.$

The solutions λ to this equation are the eigenvalues of A.

🔑 *Definition: Eigenvector:*

A non-zero vector $\mathbf{v}$ that satisfies the above equation for a given eigenvalue λ.

For ***each eigenvalue*** λ_i, we can find its corresponding eigenvector $\mathbf{v}_i$ by solving the linear system:

$(A - \lambda_i I)\mathbf{v}_i = 0;$

This system represents the null space of the matrix $(A - \lambda_i I)$, which gives us the eigenvectors associated with the eigenvalue λ_i.

🔍 *Example*

Consider the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Step 1: Find Eigenvalues:

To find the eigenvalues, we solve the characteristic equation:

$$\det(A - \lambda I) = 0.$$

where I is the identity matrix:

Now we calculate the determinant:

$$\det(A - \lambda I) = (2 - \lambda)(2 - \lambda) - (1)(1) = (2 - \lambda)^2 - 1.$$

Expanding this gives the quadratic equation:

$$(\lambda - 3)(\lambda - 1) = 0.$$

Thus, the eigenvalues are:

$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = 1.$$

Step 2: Find Corresponding Eigenvectors:

Now, we find the eigenvectors $\mathbf{v}_i$ corresponding to each eigenvalue.

- For $\lambda_1 = 3. \mathbf{v}_1 = (v_1, v_2) \in \mathbb{R}^2$

Substituting λ_1 into $A - \lambda I$:

$$A - 3I = \begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We solve the equation:

$$(A - 3I)\mathbf{v}_1 = 0.$$

This leads to the following system of equations:

$$\begin{aligned} -v_1 + v_2 &= 0 \\ v_1 - v_2 &= 0 \end{aligned}$$

From either equation, we find:

$$v_1 = v_2.$$

We have:

$$\mathbf{v}_1 = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We can choose $(v_1 = 1)$:

$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

- For $\lambda_2 = 1$

Substituting λ_2 into $A - \lambda I$:

$A - 1I = \begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$

We solve the equation:

$(A - 1I)\mathbf{v}_2 = 0.$ such $\mathbf{v}_2 = (x_1, x_2) \in \mathbb{R}^2$

This leads to the following system of equations:

$x_1 + x_2 = 0$

$x_1 + x_2 = 0$

From either equation, we find:

$x_= - x_2.$

$\mathbf{v}_2 = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

We can choose $x_2 = 1$:

$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

1.4. Diagonalization of Matrices

📎 *Note:Conditions for Diagonalization*

A square matrix A is said to be ***diagonalizable*** if there exists a ***diagonal matrix*** D and an ***invertible matrix*** P such that:

$A = PDP^{-1}.$

The necessary and sufficient conditions for A to be diagonalizable are:

- A has ***n linearly independent eigenvectors*** where n is the size of the matrix.

Similarity Transformations: If two matrices A and B are similar, there exists an invertible matrix P such that:

$B = PAP^{-1}$

- Similar matrices represent ***the same*** linear transformation in different bases.
- The geometric multiplicity (number of linearly independent eigenvectors) of each eigenvalue of A must equal its algebraic multiplicity (the power of each eigenvalue in the characteristic polynomial).

Relationship Between Eigenvalues and Diagonalizability

The relationship between eigenvalues and diagonalizability is as follows:

- If all the eigenvalues of A are ***distinct***, then A is diagonalizable.
- If A has ***repeated eigenvalues***, it may or may not be diagonalizable. Diagonalizability depends on whether there are ***enough linearly independent eigenvectors*** to form the matrix P.

Method:Diagonalization Process:

- ***Find Eigenvalues:*** Solve the characteristic polynomial to find the eigenvalues.
- ***Find Eigenvectors:*** For each eigenvalue, solve $(A - \lambda I)\mathbf{v} = 0$ to find the corresponding eigenvectors.
- ***Form Matrix P:*** Construct matrix P using the eigenvectors as columns.
- ***Form Matrix D:*** Construct the diagonal matrix D with the eigenvalues on the diagonal.
- ***To verify:*** Check if $A = PDP^{-1}$.

Example:From the example above

Step 3: Form the Matrices P and D:

Now we can form the matrix P of eigenvectors and the diagonal matrix D of eigenvalues:

$$P = [\mathbf{v}_1 \mid \mathbf{v}_2] = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

$$D = \mathrm{diag}(3, 1) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$