Exercises solution

> **Exercise** n°1

1. Matrix representation of T :

To find the matrix A of T, we apply T to the standard basis vectors of $\mathbb{R}^3$:

Thus, the matrix representation of T is:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

2. Eigenvalues of T :

The eigenvalues are found by solving the characteristic equation:

$$\det(A - \lambda I) = 0,$$

where I is the 3×3 identity matrix.

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix}.$$

The determinant is:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix}.$$

Expanding along the first row:

$$\det(A - \lambda I) = (2-\lambda)\det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}.$$

$$\det(A - \lambda I) = (2-\lambda)((1-\lambda)(1-\lambda)).$$

$$\det(A - \lambda I) = (2-\lambda)(1-\lambda)^2.$$

The eigenvalues are:

$$\lambda_1 = 2, \quad \lambda_2 = 1 \quad \text{(with multiplicity 2)}.$$

REMARK : since the matrix A is triangular matrix so we can directly deduce that $\lambda_1 = 2, \quad \lambda_2 = \lambda_3 = 1$.

3. Eigenvectors

For $\lambda = 2$:

Solve $(A - 2I)v = 0$:

$$A - 2I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

The system reduces to:

$$x_2 = x_3, \quad x_1 = 0.$$

An eigenvector for $\lambda = 2$ is:

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda = 1$:

Solve $(A - I)v = 0$:

$$A - I = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

This reduces to:

$$x_1 = 0.$$

Two linearly independent eigenvectors for $\lambda = 1$ are:

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

4. ***Image of*** $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1+1 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

The image is:

$$T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

> **Exercise** n°2

1. Covariance Matrix:

The covariance matrix is a symmetric matrix that shows the variances along the diagonal and the covariances in the off-diagonal elements. From the given data:

$$\Sigma = \begin{bmatrix} \text{Var}(X) & \text{Cov}(X,Y) & \text{Cov}(X,Z) \\ \text{Cov}(X,Y) & \text{Var}(Y) & \text{Cov}(Y,Z) \\ \text{Cov}(X,Z) & \text{Cov}(Y,Z) & \text{Var}(Z) \end{bmatrix}$$

$$\Sigma = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

2. Finding the Eigenvalues

The characteristic polynomial $\mathcal{P}_V(\lambda)$ is given by:

$$\mathcal{P}_V(\lambda) = \det(V - \lambda I) = 0$$

The matrix $V - \lambda I$ is:

$$V - \lambda I = \begin{pmatrix} 4-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 4-\lambda \end{pmatrix}$$

The determinant of this matrix is:

$$\mathcal{P}_V(\lambda) = (4-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} - 0 = (4-\lambda)\,[(2-\lambda)(4-\lambda) - 1] = 0;$$

Simplifying further:

$$\mathcal{P}_V(\lambda) = (4-\lambda)\,[(2-\lambda)(4-\lambda) - 2] = 0$$

$$\mathcal{P}_V(\lambda) = (4-\lambda)\left[\lambda^2 - 6\lambda + 6\right] = 0$$

Thus, we have two factors to solve for eigenvalues:

$$4 - \lambda = 0 \quad \Rightarrow \quad \lambda = 4$$

3. Solving the quadratic equation $\lambda^2 - 6\lambda + 6 = 0$:

$$\Delta = (-6)^2 - 4 \times 1 \times 6 = 36 - 24 = 12$$

$$\lambda = \frac{-(-6) \pm \sqrt{12}}{2 \times 1} = \frac{6 \pm \sqrt{12}}{2}$$

$$\lambda_1 = 4.73 \quad \lambda_2 = 1.27$$

***Step 2:** Eigenvalues, Percentages, and Cumulative Percentages*

Now, let's organize the eigenvalues and their corresponding percentages of variance:

$$\text{Percentage }(\%) = \frac{\text{Eigenvalue}}{\text{Total Variance}} \times 100.$$

$$\text{Cumulative Percentage }(\%) = \sum_{i=1}^{k} \text{Percentage }(\%)_i$$

$$\text{where Total Variance} = \sum_{i=1}^{n} \text{Eigenvalue}_i.$$

Eigenvalue	Percentage of Variance	Cumulative Percentage
4.732	47.321%	47.321%
4.000	40.000%	87.321%
1.268	12.679%	100.000%

4. Percentage of Variance Explained by Each Principal Component :

First principal component: 47.32%

Second principal component: 40.00%

Third principal component: 12.679%

5. Kaiser Criterion:

According to the Kaiser criterion, all three principal components should be retained because all eigenvalues are greater than 1.

6. Eigenvalue Equation

$$(V - \lambda_2 I)\, u_2 = 0$$

The system of equations becomes:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$y = 0 \quad \text{(from the first equation)}$

$x - 2y + z = 0 \quad \Rightarrow \quad x + z = 0 \quad \Rightarrow \quad x = -z \quad \text{(from the second equation)}$

The solution to the system is:

$$\boldsymbol{u}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix}$$

We can choose $z = 1$, so the eigenvector becomes:

$$\boldsymbol{u}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

7. Normalize the eigenvector

$$\|\mathbf{u}_2\| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

To normalize the vector, divide each component by the magnitude:

$$\mathbf{u}_2 = \left(\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)^t$$

This simplifies to:

$$\mathbf{u}_2 = \begin{pmatrix} -0.71 \\ 0 \\ 0.71 \end{pmatrix} \quad \text{(normalized eigenvector)}$$

> Exercise n°3:

1. The Centered Matrix M:

$$M = \begin{pmatrix} -1 & m_{12} & m_{13} \\ 0 & 0 & 0 \\ m_{31} & -1 & m_{33} \\ m_{41} & m_{42} & 0 \end{pmatrix}$$

We know that: $F_1 = MU_1$ and $F_2 = MU_2$

$$F_1 = \begin{pmatrix} -1 & m_{12} & m_{13} \\ 0 & 0 & 0 \\ m_{31} & -1 & m_{33} \\ m_{41} & m_{42} & 0 \end{pmatrix} \begin{pmatrix} 0.46 \\ 0.88 \\ -0.08 \end{pmatrix} = \begin{pmatrix} -1.42 \\ 0 \\ -0.80 \\ 2.22 \end{pmatrix}$$

$$F_2 = \begin{pmatrix} -1 & m_{12} & m_{13} \\ 0 & 0 & 0 \\ m_{31} & -1 & m_{33} \\ m_{41} & m_{42} & 0 \end{pmatrix} \begin{pmatrix} -0.37 \\ 0.27 \\ 0.89 \end{pmatrix} = \begin{pmatrix} 0.99 \\ 0 \\ -1.16 \\ 0.17 \end{pmatrix}$$

Thus, for the equation:

By solving the system of equations, we get:

$$\Rightarrow \begin{cases} 0.88m_{12} - 0.08m_{13} = -1.42 + 0.46 & (1) \\ 0.27m_{12} + 0.89m_{13} = 0.99 - 0.37 & (2) \end{cases}$$

$$\Rightarrow \begin{cases} 0.88m_{12} - 0.08m_{13} = -0.96 \\ 0.27m_{12} + 0.89m_{13} = 0.62 \end{cases}$$

$$\Rightarrow \begin{cases} 0.88m_{12} = -0.96 + 0.08m_{13} \\ 0.27m_{12} + 0.89m_{13} = 0.62 \end{cases} \Rightarrow \begin{cases} m_{12} = \dfrac{-0.96 + 0.08m_{13}}{0.88} \\ 0.27m_{12} + 0.89m_{13} = 0.62 \end{cases}$$

$$\Rightarrow \begin{cases} m_{12} = -1.09 + 0.09m_{13} \\ 0.27m_{12} + 0.89m_{13} = 0.62 \end{cases}$$

Substituting equation (1) into equation (2), we get:

$$0.27(-1.09 + 0.09m_{13}) + 0.89m_{13} = 0.62 \Rightarrow -0.29 + 0.02m_{13} + 0.89m_{13} = 0.62 \Rightarrow$$
$$0.91m_{13} = 0.91 \Rightarrow m_{13} = \frac{0.91}{0.91} \Rightarrow m_{13} = 1$$

By substituting into equation (1), we find:

$$-0.46 + 0.88m_{12} - 0.08(1) = -1.42 \Rightarrow 0.88m_{12} = -0.88 \Rightarrow m_{12} = \frac{-0.88}{0.88} \Rightarrow m_{12} = -1$$

We are given the matrix:

$$M = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$$

2. The Basic Data Matrix:

We know that:

$$M = X - 1_n g'$$
$$\Leftrightarrow X = M + 1_n g'$$
$$\Leftrightarrow X = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Therefore:

$$X = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 4 & 6 \\ 3 & 4 & 6 \\ 3 & 4 & 6 \\ 4 & 4 & 6 \end{pmatrix} \Leftrightarrow X = \begin{pmatrix} 2 & 3 & 7 \\ 3 & 4 & 6 \\ 3 & 3 & 5 \\ 4 & 6 & 6 \end{pmatrix}$$

3. Variance-Covariance Matrix:

$$V = \frac{1}{4}X'X = \frac{1}{4}\begin{pmatrix} -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 2 \\ 1 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix} \Rightarrow V = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} & -\frac{1}{4} \\ \frac{3}{4} & \frac{3}{2} & 0 \\ -\frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}$$

4. Total Variance:

$$I = \sum_{i=1}^{\alpha} \lambda_\alpha = \text{trac}(V)$$

$$I = \frac{1}{2} + \frac{3}{2} + \frac{1}{2}$$

$$I = 2.5$$

5. Remaining Eigenvalue:

We have:

$\text{trac}(V) = \sum_{i=1}^{\alpha} \lambda_\alpha$

From this, we can deduce the remaining eigenvalue as follows:

$$\text{trac}(V) = \left(\frac{1}{2} + \frac{3}{2} + \frac{1}{2}\right) = 2.5 \Leftrightarrow \sum_{i=1}^{\alpha} \lambda_\alpha = \lambda_1 + \lambda_2 + \lambda_3 = 2.5 \Leftrightarrow \lambda_1 = (2.5 - 0.66) = 1.84$$