EXPECTED UTILITY THEORY

Dr. L. OUDAIFIA

Plan

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in a certain world, people like to maximize utility. In a world of uncertainty, it seems intuitive that individuals would maximize ***expected utility***. This refers to a construct used to explain the level of satisfaction a person gets when faced with uncertain choices. The intuition is straightforward, proving it axiomatically was a very challenging task. John von Neumann and Oskar Morgenstern (1944) advocated an approach that leads us to a formal mathematical representation of maximization of expected utility.

In 1944, John Von Neumann and Oskar Morgenstern published their book, *Theory of Games and Economic Behavior*. In this book, they moved on from Bernoulli's formulation of a utlity function over wealth, and defined an *expected utility function* over *lotteries*, or gambles. Theirs is an axiomatic derivation, meaning, a set of assumptions over people's preferences is required before one can construct a utility function.

I. The utility theory

This notion that an individual derives satisfaction from wealth seems to work more often than not in economic situations. The economic theory that links the level of satisfaction to a person's wealth level, and thus to consumption levels, is called utility theory. Its basis revolves around individuals' preferences, but we must use caution as we apply utility theory. The utility theory is utilized to compare two or more options. Thus, by its very nature, we refer to the utility theory as an "ordinal" theory, which rank orders choices, rather than "cardinal" utility, which has the ability to attach a number to even a single outcome where there are no choices involved.

II. Historical Motivation: Resolution of Bernoulli Paradox (Machina, 1987)

During the development of modern probability theory in the 17th century, mathematicians such as Blaise Pascal and Pierre de Fermat assumed that the attractiveness of a gamble offering the payoffs (x1, x2, . . . , xn) with probability (p1, p2, . . . , pn) was given by its expected value

$$\bar{x} = \sum_{i=1}^{n} p_i x_i.$$

The fact that individuals consider more than just expected value, however, was dramatically illustrated by an example posed by Nicholas Bernoulli in 1728 and now known as the St.Petersburg Paradox:

• Suppose someone offers to toss a fair coin repeatedly until it comes up heads and to pay you $1 if this happens on the first toss, $2 if it takes two tosses to land a head, $4 if it takes three tosses, $8 if it takes four tosses, etc.. What is the largest sure gain you would be willing to forgo in order to undertake a single play of this game?

Since this gamble offers a 1/2 chance of winning \$1, a 1/4 chance of winning \$2, etc., its expected value is

$$\frac{1}{2} \times \$1 + \frac{1}{4} \times \$2 + \frac{1}{8} \times \$4 + \ldots = \frac{1}{2} + \frac{1}{2} + \ldots = \infty.$$

So it should be preferred to any finite sure gain. However, it is clear that few people would forgo more than a moderate amount for a one-shot play.

The resolution of this paradox was proposed by Gabriel Cramer and Nicholas's cousin Dankiel Bernoulli. They argued that a gain of $200 was not necessarily "worth" twice as much as a gain of $100. This suggested that people don't use expected value to evaluate gamble, rather they use some sort of "expected utility function" to do so.

The expected utility function (or termed von-Neumann-Morgenstern utility) is defined as

$$V = \sum_{i=1}^{n} p_i U(x_i).$$

Again, note that expected utility function is not unique, but several functions can model the preferences of the same individual over a given set of uncertain choices or games. What matters is that such a function (which reflects an individual’s preferences over uncertain games) exists. The expected utility theory then says if the axioms provided by von Neumann-Morgenstern are satisfied, then the individuals behave as if they were trying to maximize the expected utility.

The most important insight of the theory is that the expected value of the dollar outcomes may provide a ranking of choices different from those given by expected utility. The **expected utility theory** then says persons shall choose an option that maximizes their expected utility rather than the expected wealth.

III. Basic Concept

At its core, Expected Utility Theory assumes that individuals make decisions by maximizing their expected utility, which is a mathematical expectation of the utility (satisfaction or value) they receive from various possible outcomes, weighted by the probability of each outcome occurring.

- **Utility:** Utility is a measure of satisfaction or value derived from a particular outcome. It is subjective and can vary from person to person.
- **Expected utility:** The expected utility is the sum of the utilities of all possible outcomes, each weighted by the probability of that outcome happening.

The expected utility of a choice (or decision) can be represented as:

$$EU = \sum_{i=1}^{n} P_i \times U(O_i)$$

Where:

- EU = Expected utility of the decision.
- P_i = Probability of the i-th outcome occurring.
- $U(O_i)$ = Utility (or value) of the i-th outcome.
- The summation goes over all possible outcomes n.

Example

Imagine you're deciding between two investment options, each with different potential returns:

- Investment A: 80% chance to earn $100 and 20% chance to lose $50.
- Investment B: 50% chance to earn $150 and 50% chance to lose $30.

To apply Expected Utility Theory, you'd assign utility values to the amounts of money you might receive (which could be simple or based on your personal risk preferences) and calculate the expected utility for each investment.

Let’s say you assign the following utility values (assuming linear utility for simplicity):

- Utility of \$100 = 100
- Utility of \$150 = 150
- Utility of -\$50 = -50
- Utility of -\$30 = -30

Now, calculate the expected utilities:

For **Investment A**:

$$EU(A) = (0.8 \times 100) + (0.2 \times -50) = 80 - 10 = 70$$

For **Investment B**:

$$EU(B) = (0.5 \times 150) + (0.5 \times -30) = 75 - 15 = 60$$

According to Expected Utility Theory, you would choose **Investment A**, as it has the higher expected utility (70 vs. 60).

VI. Key Assumptions

Rationality: The decision-maker is assumed to be rational and aims to maximize expected utility.

Risk Aversion or Seeking: People can exhibit different attitudes toward risk. If someone prefers a certain outcome over a gamble with the same expected value, they are considered risk-averse. If they prefer riskier options, they are risk-seeking.

Consistent Preferences: The theory assumes that an individual's preferences over different outcomes are consistent and stable over time.

V. Limitations

While Expected Utility Theory is a widely used model in economics and decision-making, it has been criticized for not fully explaining real human behavior in some cases. For example:

Behavioral anomalies: People often make decisions that contradict the predictions of Expected Utility Theory, such as the endowment effect (valuing something they own more than its market value) or loss aversion (feeling the pain of losses more intensely than the pleasure of gains).

Non-linear utility: In reality, utility may not always be linear (as the example assumed), and people may have diminishing or increasing marginal utility for money or outcomes.

In these cases, alternatives like Prospect Theory (which incorporates psychological aspects of decision-making) have been developed to address these inconsistencies.

IV. Different Risk Attitudes

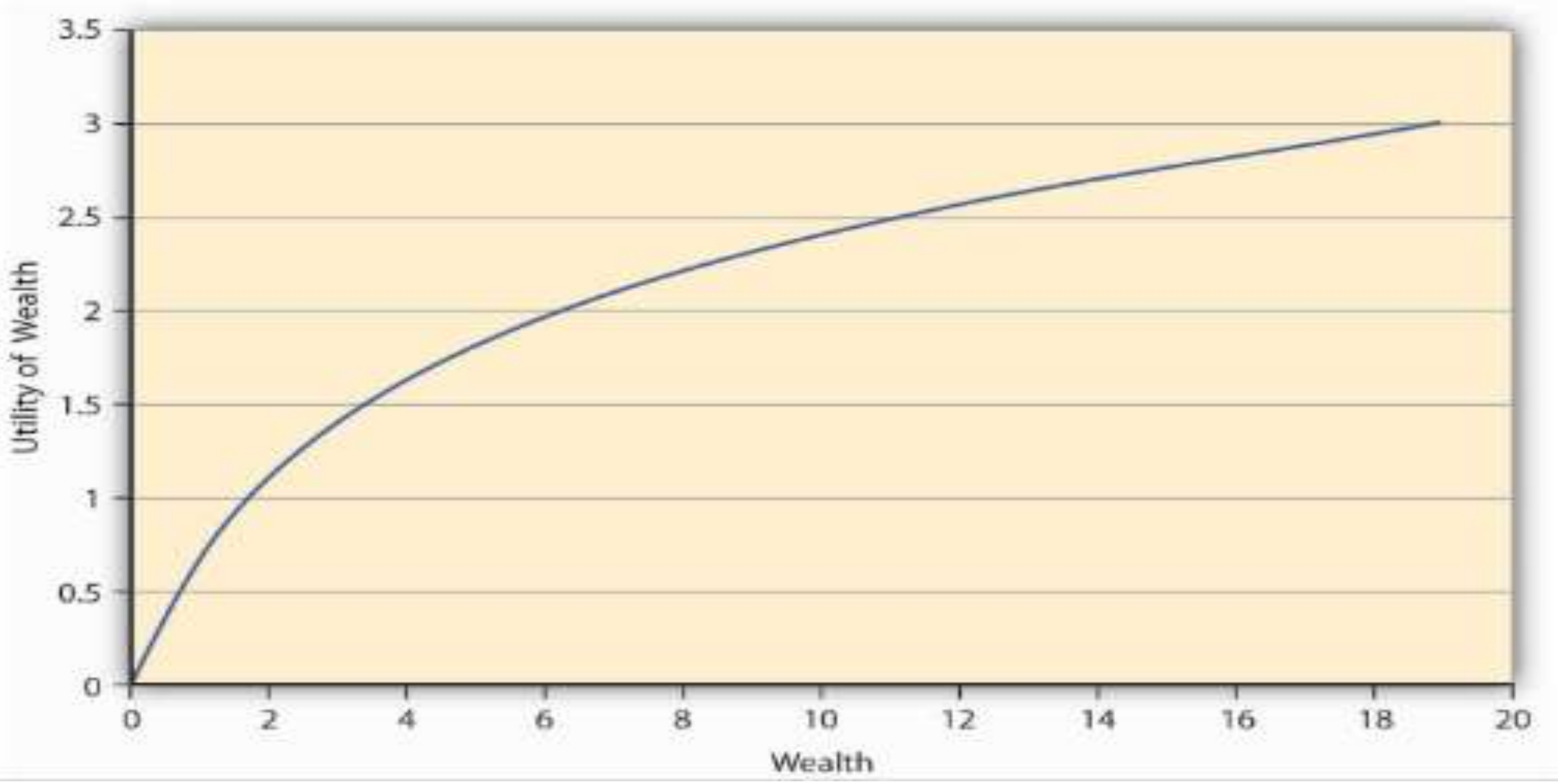
We can group people's attitudes towards risk into 3 distinct categories, based on the form of their respective ***Bernoulli utility functions***. Let's illustrate this with a simple investment, which pays $10 in the first situation, and $20 in the second situation. The expected value of this investment is, of course: (0.5 * 10) + (0.5 * 20) = $15.

1. Risk Averse:

A person who is risk averse prefers to avoid uncertainty and is willing to sacrifice some potential return in order to reduce or eliminate risk. They generally prefer safer, more predictable outcomes. For example, they might choose a stable investment, even if it offers a lower return, because they want to avoid the possibility of losing money.

If a person's utility of the expected value of an investment is greater than their expected utility from the investment itself, they are said to be risk-averse. This is a more precise definition of ***Bernoulli's idea***. Risk-averse behavior is captured by a concave Bernoulli utility function, like a logarithmic function. For the above investment, a risk-averse person whose Bernoulli utility function took the form $u(w) = \log(w)$, where w was the outcome, would have an expected utility over the investment of :$0.5 * \log(10) + 0.5 * \log(20) = 1.15$,while their utility of the expected value of the investment is $\log(15) = 1.176$.

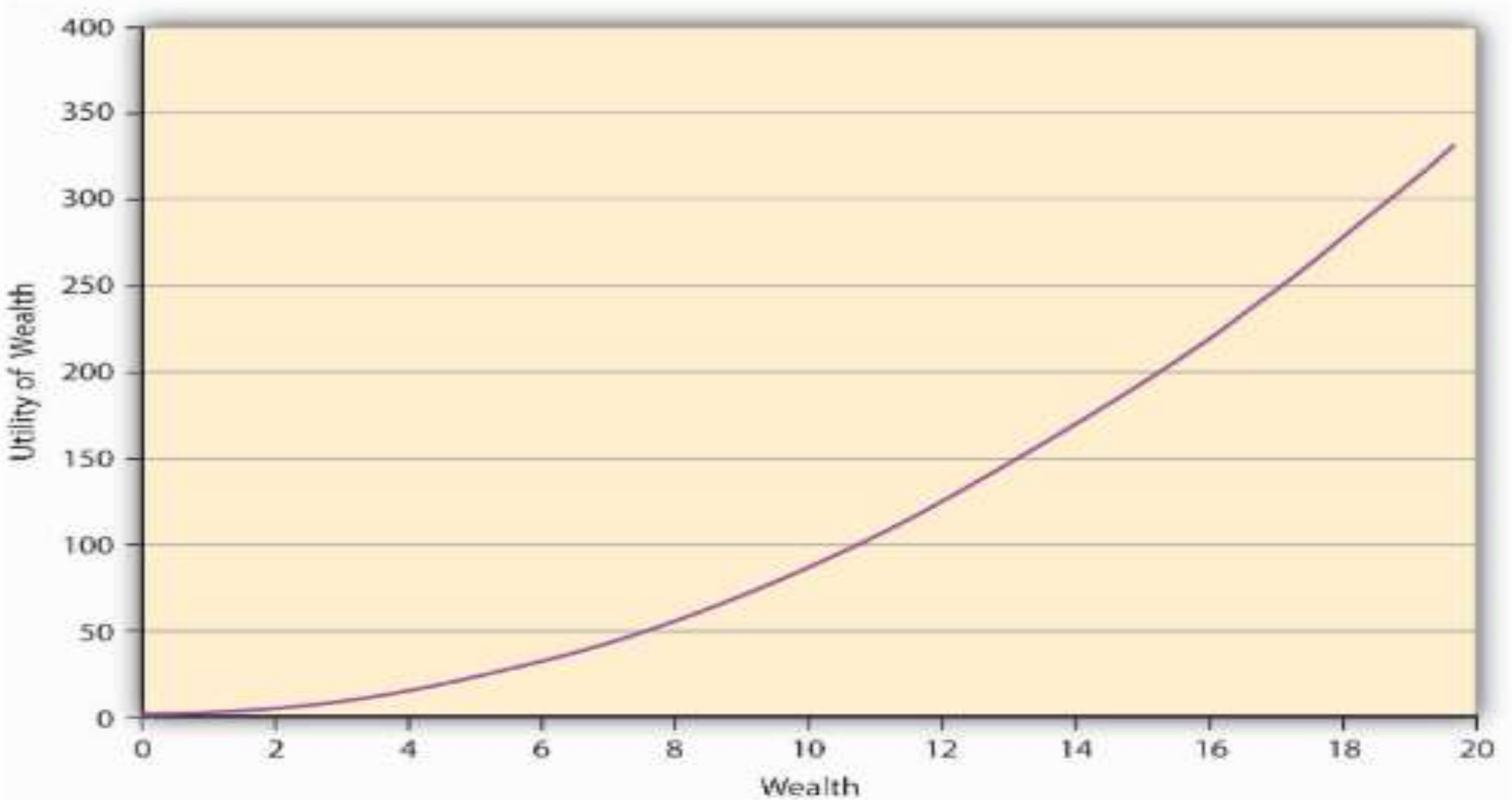
Figure 3.2 A Utility Function for a Risk-Averse Individual



2. Risk Seeking: A person who is risk seeking is willing to take on higher levels of uncertainty in exchange for the potential for higher rewards. They tend to prefer situations where there is more risk, even if it means a greater chance of loss. An example of this would be someone who invests in highly speculative stocks or gambles, hoping to make large gains, despite the high likelihood of losing.

For the above investment, a risk-loving person whose Bernoulli utility function took the form u(w) = w2 would have an expected utility over the investment of: 0.5 * 102 + 0.5 * 202 = 250, while their utility of the expected value of the investment is 152 = 225.

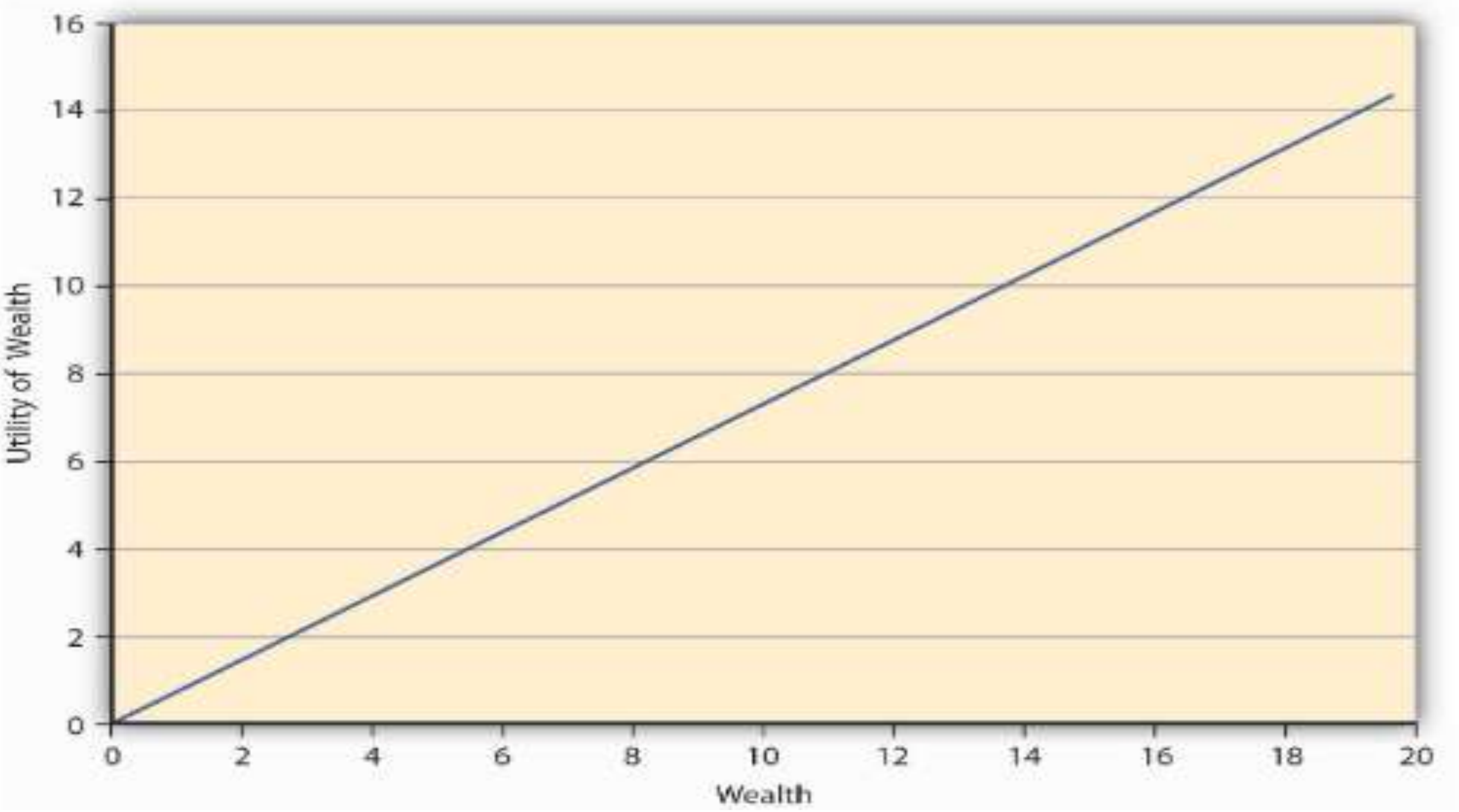
Figure 3.3 A Utility Function for a Risk-Seeking Individual



3. Risk Neutral: A person who is risk neutral is indifferent to risk. They don't have a preference for taking on risk or avoiding it. They care only about the expected outcome and are willing to accept risks as long as the expected return is equal to or greater than the risk. For example, if two investments have the same expected value, they would treat them equally, regardless of how risky each option is.

For the above investment, a risk-neutral person whose Bernoulli utility function took the form $u(w) = 2w$ would have an expected utility over the investment of: $(0.5 * 2 * 10) + (0.5 * 2 * 20) = 30$, while their utility of the expected value of the investment is $2 * 15 = 30$.

Figure 3.4 A Utility Function for a Risk-Neutral Individual

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