MATRICES

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Throughout the chapter, $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$, and n and m are two nonzero natural numbers.

Matrices

Definitions

A matrix A of size $(m;n)$, with m rows and n columns, is a rectangular array of elements from $\mathbb{K}$:

- The numbers in the array are called the coefficients of A.
- The coefficient located at the $i-th$ row and $j-th$ column is denoted by a_{ij}.
- Such an array is represented as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{i,} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}, \quad \text{ou} \quad A = (a_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}}, \text{ ou } \ A = (a_{ij})_{m,n}.$$

Matrices

Example

We say that the matrix A is of size $m \times n$ (read as "m by n") (respecting the reading order).

$$A = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 3 & 7 \end{pmatrix}$$

It is a 2×3 matrix with, for example, $a_{11} = 1$ and $a_{23} = 7$.

Special matrices

Zero Matrix

A matrix where all of its elements are zero. It is denoted as $0_{m \times n}$.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Square matrices

Square matrices are matrices in which the number of rows and columns are equal. This number of rows and columns is called the **order** of the matrix.

Example

A matrix of order 3 (a 3×3 matrix) is:

$$\begin{pmatrix} 1 & 6 & -3 \\ 2 & 0 & 4 \\ 5 & 0 & -1 \end{pmatrix}$$

The coefficients that have the same row and column indices are called the diagonal coefficients.

Special matrices

Lower triangular matrices

Lower triangular matrices are square matrices in which all the coefficients strictly above the diagonal are zero.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Upper triangular matrices

Upper triangular matrices are square matrices in which all the coefficients strictly below the diagonal are zero.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

Special matrices

The identity matrix

The identity matrix is the diagonal matrix in which all the diagonal elements are equal to 1. It is denoted as I_n, the identity matrix of order n.

Example The identity matrix of order 3

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Diagonal matrices

Diagonal matrices are square matrices that are both upper triangular and lower triangular at the same time. The only nonzero elements are those on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

Operations on Matrices

Addition of Matrices

The sum of two matrices A and B, of the same size $m \times n$, is defined as:

$$c_{ij} = a_{ij} + b_{ij}.$$

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 2 & 0 \\ 4 & 7 & 1 \end{pmatrix},$$

$$A + B = \begin{pmatrix} 6 & 4 & 3 \\ 4 & 8 & 1 \end{pmatrix}.$$

Operations on Matrices

Proposition

Matrix Addition is Associative $(A+B)+C = A+(B+C)\,.$

Matrix Addition is Commutative $A+B = B+A.$

Multiplication of a Matrix by a Scalar

Let $A = (a_{ij})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$ and $\lambda \in \Bbbk,\ \ \lambda A = (\lambda a_{ij})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}\,.$

Example

If $\lambda = 2,$ and $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix}$ so $\lambda A = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 0 \end{pmatrix}.$

Operations on Matrices

Propositions

- $\lambda (A + B) = \lambda A + \lambda B.$
- $(\lambda + \mu) A = \lambda A + \mu A.$
- $(\lambda\mu) A = \lambda (\mu A).$
- $1A = A.$

Operations on Matrices

Matrix Multiplication

The product of two matrices is defined only when the number of columns in the first matrix is equal to the number of rows in the second matrix.

Let $A = (a_{ij})_{m,n}$ and $B = (b_{ij})_{n,p}$,the product $A \times B = (c_{ij})_{m,p}$ is a matrix of size (m,p) such that $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

Example

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 0 & 5 \\ 1 & 7 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 5 \\ 1 & 2 \\ 0 & 4 \end{pmatrix}.$$

$$A \times B = \begin{pmatrix} 2\times 4 + 1\times 1 + 4\times 0 & 2\times 5 + 1\times 2 + 4\times 4 \\ 3\times 4 + 0\times 1 + 5\times 0 & 3\times 5 + 0\times 2 + 5\times 4 \\ 1\times 4 + 7\times 1 + 6\times 0 & 1\times 5 + 7\times 2 + 6\times 4 \end{pmatrix} = \begin{pmatrix} 9 & 28 \\ 12 & 35 \\ 11 & 43 \end{pmatrix}.$$

Operations on Matrices

Pitfalls to Avoid:

- Matrix multiplication is not commutative in general. $AB \neq BA.$

Example

$$\begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 14 & 3 \\ -2 & -6 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 10 & 2 \\ 29 & -2 \end{pmatrix}.$$

- $AB = 0$ does not imply $A = 0$ or $B = 0.$

Example

$$A = \begin{pmatrix} 0 & -1 \\ 0 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Operations on Matrices

Properties

- Associativity

$$(AB)\,C = A\,(BC)\,.$$

- Role of Identity Matrices

$$AI_n = A \quad et \quad I_m A = A.$$

- Distributivity with respect to addition.

$$(A + B)\,C = AC + BC \qquad A\,(B + C) = AB + AC.$$

- Compatibility with scalar multiplication.

$$\lambda\,(AB) = (\lambda A)\,B = A\,(\lambda B)\,.$$

Operations on Matrices

Power of a Matrix

The power of a matrix refers to the repeated multiplication of a **square matrix** by itself.

$$A^p = \underbrace{A \times A \times \ldots \times A}_{p\ facteurs}.$$

$$A^{p+1} = A^p \times A,$$

Example

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

$$A^2 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

$$A^3 = A^2 \times A = \begin{pmatrix} 1 & 0 & 7 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{pmatrix},$$

$$A^0 = I$$

Operations on Matrices

Transpose of a Matrix

The **transpose** of a matrix is obtained by swapping its rows and columns.

If A is a matrix of size $m \times n$, its **transpose**, denoted as A^T, is a matrix of size $n \times m$, where:

$$^tA = (a_{ji})_{\substack{1\leq j\leq n \\ 1\leq i\leq m}}.$$

Proposition

- $^t(^tA) = A.$
- $^t(\lambda A) = \lambda\, ^tA.$
- $^t(A+B) = {}^tA + {}^tB.$
- $^t(AB) = {}^tB\, ^tA.$

Operations on Matrices

The trace of a matrix

the **trace** of a matrix A, denoted as $Tr(A)$, is the sum of the diagonal elements of the matrix.
Formally, if $A = (a_{ij})$ is a square matrix of size $n \times n$, then:

$$trA = a_{11} + a_{22} + \ldots + a_{nn}.$$

Example

If $A = \begin{pmatrix} 1 & 1 & 2 \\ 5 & 2 & 8 \\ 11 & 0 & -10 \end{pmatrix}$ then $trA = 1 + 2 + (-10) = -7.$

Proposition

- $tr\,(A + B) = trA + trB.$
- $tr\,(\alpha A) = \alpha trA$
- $tr\,({}^tA) = trA.$
- $tr\,(AB) = tr\,(BA).$

Operations on Matrices

The determinant of a matrix

The **determinant of a matrix** is a scalar value that provides important information about a square matrix. It is used in solving linear equations, finding the inverse of a matrix, and determining properties such as whether a matrix is invertible.

For an $n \times n$ matrix $A = [a_{ij}]$, the determinant is denoted as $\det(A)$ or $|A|$.

1. **For a 1×1 matrix** $A = [a]$, the determinant is simply: $\det(A) = a$

2. For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the determinant is calculated as: $\det(A) = ad - bc$

3. For a 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ the determinant is given by: $\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$

Operations on Matrices

The determinant of a matrix

General Formula for $n \times n$ Matrices

For larger matrices, the determinant is computed using **cofactor expansion** (Laplace expansion):

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

where M_{ij} is the determinant of the **minor matrix** obtained by deleting the ith row and jth column of A. This process can be applied recursively.

Example

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 0 & 5 \\ 1 & 7 & 6 \end{pmatrix}.$$

$$\det(A) = \begin{vmatrix} 2 & 1 & 4 \\ 3 & 0 & 5 \\ 1 & 7 & 6 \end{vmatrix} = 2\begin{vmatrix} 0 & 5 \\ 7 & 6 \end{vmatrix} - 1\begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} + 4\begin{vmatrix} 3 & 0 \\ 1 & 7 \end{vmatrix},$$

$$= 2 \times (-35) - 1 \times 13 + 4 \times 21 = 1.$$

Operations on Matrices

Inverse of a Matrix

Let A be a square matrix of size $n \times n$. If there exists a square matrix B of size $n \times n$ such that:

$$AB = I \quad et \quad BA = I,$$

where I is the identity matrix, then A is called **invertible**.

- B is called the **inverse of** A and is denoted as A^{-1}.

Example Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

Study whether A is invertible means examining the existence of a matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $AB = I$ and $BA = I$.

Operations on Matrices

$$AB = I \iff \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\iff \begin{pmatrix} a+2c & b+2d \\ 3c & 3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This equality is equivalent to the system:

$$\begin{cases} a + 2c = 1 \\ b + 2d = 0 \\ 3c = 0 \\ 3d = 1 \end{cases} . \implies B = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}.$$

so

$$A^{-1} = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}.$$

Note

Matrices 2×2: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Operations on Matrices

Inverse of a Matrix

The adjoint matrix (or adjugate matrix)

The **adjoint matrix** (or **adjugate matrix**) of a square matrix A is the transpose of its cofactor matrix. It is useful in calculating the inverse of a matrix and in other linear algebra applications.

For an $n \times n$ matrix A, the **adjugate matrix**, denoted as $adj\ (A)$, is given by:

$$\mathrm{adj}(A) = \mathrm{Cof}(A)^T$$

where:

- $\boldsymbol{Cof(A)}$ is the **cofactor matrix**, whose elements are the cofactors of A.
- The ***cofactor*** of an element a_{ij} is given by:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the determinant of the **minor matrix** obtained by deleting the $i-th$ row and $j-th$ column from A.

Operations on Matrices

Inverse of a Matrix

The adjoint matrix (or adjugate matrix)

Relation to the Inverse: If A is an invertible matrix, then:

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}$$

Example

compute the **inverse** of matrix A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 7 & 8 & 6 \end{bmatrix}$$

compute $\det(A)$:

$$\det(A) = (1 \times (-16)) - (2 \times (-35)) + (3 \times (-28))$$
$$= -16 + 70 - 84 = -30$$

Operations on Matrices

Compute Minors and Cofactors:

1. **First row:**
   - $C_{11} = (-1)^{1+1} \det \begin{bmatrix} 4 & 5 \\ 8 & 6 \end{bmatrix} = (4 \times 6 - 5 \times 8) = (24 - 40) = -16$
   - $C_{12} = (-1)^{1+2} \det \begin{bmatrix} 0 & 5 \\ 7 & 6 \end{bmatrix} = -(0 \times 6 - 5 \times 7) = 35$
   - $C_{13} = (-1)^{1+3} \det \begin{bmatrix} 0 & 4 \\ 7 & 8 \end{bmatrix} = (0 \times 8 -$ [illegible] $\times 7) = -28$

2. **Second row:**
   - $C_{21} = (-1)^{2+1} \det \begin{bmatrix} 2 & 3 \\ 8 & 6 \end{bmatrix} = -(2 \times 6 - 3 \times 8) = 10$
   - $C_{22} = (-1)^{2+2} \det \begin{bmatrix} 1 & 3 \\ 7 & 6 \end{bmatrix} = (1 \times 6 - 3 \times 7) = -15$
   - $C_{23} = (-1)^{2+3} \det \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} = -(1 \times 8 - 2 \times 7) = 6$

3. **Third row:**
   - $C_{31} = (-1)^{3+1} \det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = (2 \times 5 - 3 \times 4) = -2$
   - $C_{32} = (-1)^{3+2} \det \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix} = -(1 \times 5 - 3 \times 0) = -5$
   - $C_{33} = (-1)^{3+3} \det \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} = (1 \times 4 - 2 \times 0) = 4$

Operations on Matrices

Thus, the **cofactor matrix** is:

$$C(A) = \begin{bmatrix} -16 & 35 & -28 \\ 10 & -15 & 6 \\ -2 & -5 & 4 \end{bmatrix}$$

so

$$\operatorname{adj}(A) = C(A)^T = \begin{bmatrix} -16 & 10 & -2 \\ 35 & -15 & -5 \\ -28 & 6 & 4 \end{bmatrix}$$

we get:

$$A^{-1} = \frac{1}{-30} \begin{bmatrix} -16 & 10 & -2 \\ 35 & -15 & -5 \\ -28 & 6 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{8}{15} & -\frac{1}{3} & \frac{1}{15} \\ -\frac{7}{6} & \frac{1}{2} & \frac{1}{6} \\ \frac{14}{15} & -\frac{1}{5} & -\frac{2}{15} \end{bmatrix}$$

Linear Transformations and Eigenvalues

Linear Transformations

A **linear transformation** is a function that maps vectors from one vector space to another while preserving both vector addition and scalar multiplication. Formally, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then for any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and scalar c, the following conditions must hold:

1. **Additivity:** $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
2. **Homogeneity:** $T(c\mathbf{u}) = cT(\mathbf{u})$.

Linear Transformations and Eigenvalues

Linear Transformations

Matrix Representation of Linear Transformations

Any linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented by an $m \times n$ matrix A. This means that for any vector $\mathbf{x} \in \mathbb{R}^n$, the transformation is given by:

$$T(\mathbf{x}) = A\mathbf{x}$$

Definition: Systems of Linear Equations

A system of linear equations can be expressed in matrix form. Consider the following system:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Linear Transformations and Eigenvalues

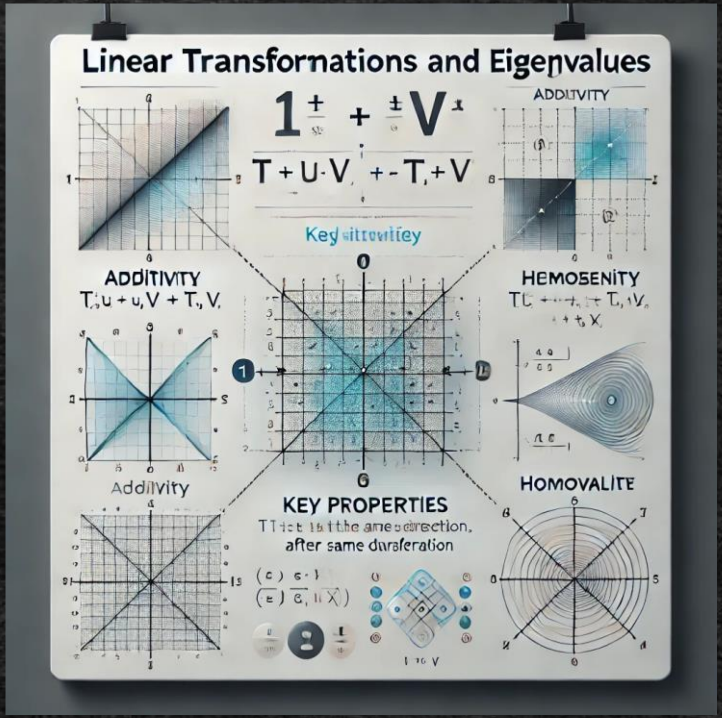
Linear Transformations

This system can be compactly written in matrix notation as:

$$AX = B$$

where:

- A is the coefficient matrix,
- X is the column vector of unknowns,
- B is the column vector of constants.



If the ***matrix A is invertible***, we can solve the system $AX = B$ by multiplying both sides by A^{-1}, giving:

$X = A^{-1}B.$

Linear Transformations and Eigenvalues

Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are key concepts in linear algebra, widely used in applications such as data science and engineering. They help simplify matrix transformations, making complex systems easier to analyze.
To compute **eigenvalues** λ, we start with the equation:

$$A\mathbf{v} = \lambda\mathbf{v}$$

Rearranging, we get:

$$(A - \lambda I)\mathbf{v} = 0$$

where I is the **identity matrix**.

For non-trivial solutions (i.e., nonzero vectors v), the determinant must be zero:

$$\det(A - \lambda I) = 0$$

This results in a **characteristic polynomial** in λ . The values of λ that satisfy this equation are called the **eigenvalues** of A.

Linear Transformations and Eigenvalues

Example

Problem

Find the eigenvalues of A, that is, find λ such that $\det(A - \lambda I) = 0$,

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}.$$

Solution

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ &= \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)(3-\lambda) - 8 \\ &= \lambda^2 - 4\lambda - 5 \end{aligned}$$

Since we want $\det(A - \lambda I) = 0$, we want $\lambda^2 - 4\lambda - 5 = 0$. This is a simple quadratic equation that is easy to factor:

$$\begin{aligned} \lambda^2 - 4\lambda - 5 &= 0 \\ (\lambda - 5)(\lambda + 1) &= 0 \\ \lambda &= -1,\ 5 \end{aligned}$$

Linear Transformations and Eigenvalues

Eigenvalues and Eigenvectors

An **eigenvector** is a nonzero vector v that satisfies the eigenvalue equation for a given eigenvalue λ.
For each eigenvalue λ_i , we determine its corresponding eigenvector v_i by solving the system:

$$(A - \lambda_i I)\mathbf{v}_i = 0$$

This system defines the null space of the matrix $(A - \lambda_i I)$, which allows us to determine the eigenvectors corresponding to the eigenvalue λ_i.

Linear Transformations and Eigenvalues

Eigenvalues and Eigenvectors

Example

Problem find the eigenvectors of the matrix A

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Solution To find the eigenvectors of the matrix A, we solve for v in the equation: $(A - \lambda I)v = 0$

The eigenvalues we found are $\lambda_1 = 5$ and $\lambda_2 = -1$.

Finding the eigenvector for $\lambda_1 = 5$:

1. Compute $A - 5I$:

$$A - 5I = \begin{bmatrix} 1-5 & 4 \\ 2 & 3-5 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix}$$

2. Solve:

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives the equation:

$$-4x + 4y = 0 \quad \Rightarrow \quad x = y$$

So, the eigenvector for $\lambda_1 = 5$ is any scalar multiple of:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Linear Transformations and Eigenvalues

Eigenvalues and Eigenvectors

Finding the eigenvector for $\lambda_2 = -1$:

1. Compute $A + I$:

$$A - (-1)I = A + I = \begin{bmatrix} 1+1 & 4 \\ 2 & 3+1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$

2. Solve:

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives the equation:

$$2x + 4y = 0 \quad \Rightarrow \quad x = -2y$$

So, the eigenvector for $\lambda_2 = -1$ is any scalar multiple of:

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

SO

- Eigenvector for $\lambda_1 = 5$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Eigenvector for $\lambda_2 = -1$: $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Linear Transformations and Eigenvalues

Diagonalization of Matrices

A square matrix A is **diagonalizable** if there exists an invertible matrix P and a diagonal matrix D such that:

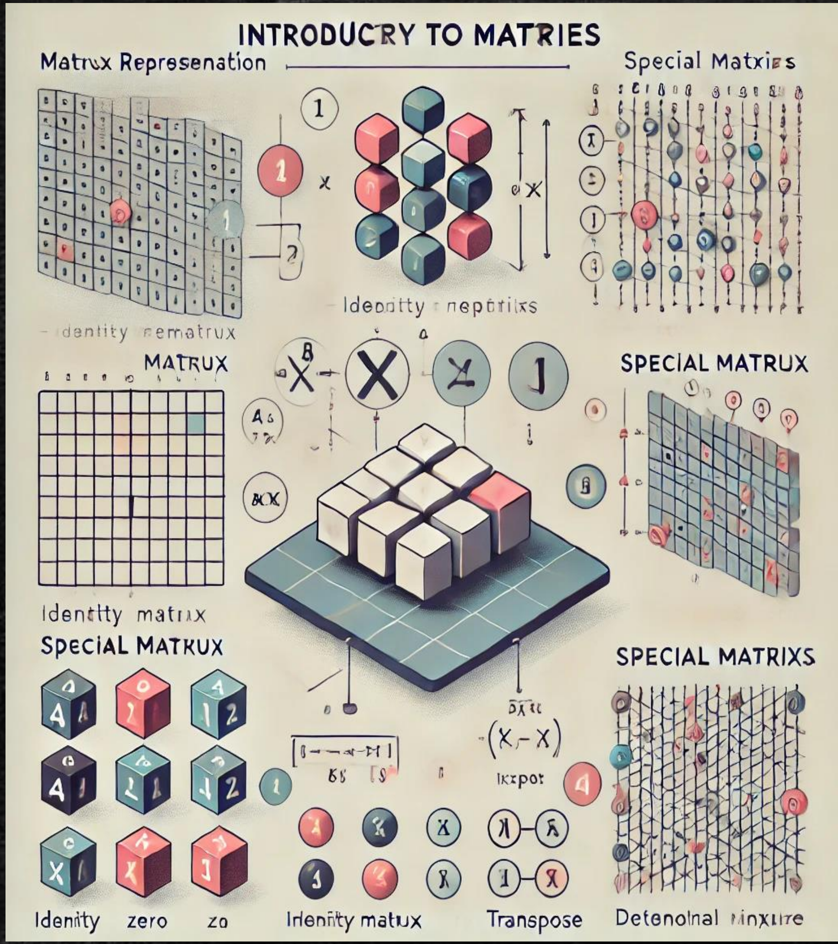
$$A = PDP^{-1}$$

Where:

- D is a diagonal matrix with the eigenvalues of A on its diagonal.
- P is a matrix whose columns are the eigenvectors of A.

Linear Transformations and Eigenvalues

Diagonalization of Matrices



Steps to Diagonalize a Matrix

1. **Find the Eigenvalues**
   Solve $\det(A - \lambda I) = 0$ to get the eigenvalues $\lambda_1, \lambda_2, \ldots$.

2. **Find the Eigenvectors**
   For each eigenvalue λ_i, solve $(A - \lambda_i I)v = 0$ to get the eigenvectors.

3. **Form the Matrix P**
   Construct P using the eigenvectors as columns.

4. **Form the Diagonal Matrix D**
   Construct D with the eigenvalues along the diagonal:

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

5. **Verify $PDP^{-1} = A$**
   Compute P^{-1} and verify that $A = PDP^{-1}$.

Linear Transformations and Eigenvalues

Diagonalization of Matrices

Example: Diagonalizing $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

We already found:

- Eigenvalues: $\lambda_1 = 5$, $\lambda_2 = -1$.
- Eigenvectors:
  - For $\lambda_1 = 5$: $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
  - For $\lambda_2 = -1$: $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Now, construct P and D:

$$P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

Find P^{-1} (using the formula for a 2×2 inverse):

$$P^{-1} = \frac{1}{(1 \times 1 - (-2) \times 1)} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

Verify:

$$PDP^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

After multiplying, we get $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, confirming that A is diagonalizable.

THANK YOU FOR YOUR ATTENTION